

THE LAMB SHIFT OF HYDROGEN AND LOW-ENERGY TESTS OF QED¹

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Abstract

Leading logarithmic corrections to the difference of Lamb shifts of s-states $E_L(1s) - 8E_L(2s)$ and to the life time of $2p_{1/2}$ state are considered. The result of Sokolov and Yakovlev for the Lamb splitting of $2s_{1/2}$ and $2p_{1/2}$ is re-evaluated and our new value is 1057.8576(21) MHz. Using value of $E_L(1s) - 8E_L(2s) = -187.237(8)$ MHz, obtained here, a new value of the Hydrogen and Deuterium ground states using all recent measurements connected with 1s or 2s Lamb shifts. The highest precision value for the Hydrogen ground state is obtained from corrected result of Sokolov and Yakovlev experiment as 8172.934(22) MHz.

The Deuterium result is $E_L(1s) - 8E_L(2s) = -187.229(8)$ MHz, using it we obtain from the Garching experiment 8183.905(224) MHz.

Some related topics are also considered. Taking into account nearest future result the fine structure constant problem are reviewed. Special attention is paid to Muonium hyperfine splitting, which it is done similar to Lamb shift calculation for and to the neutron de Broglie's wave length measurements which also could lead to some connection between the Deuterium mass and the fine structure constant.

1 Introduction

There is a lot of precision results of atomic Hydrogen spectrum, which include 1s Lamb shift, 2s or both of them. The first part of this work is devoted the problem how to obtain a connection between these two Lamb shift values. The second part of the work is devoted to problem of the fine structure constant determination and some discussion on QED calculation of corrections to the Muonium hyperfine splitting and Hydrogen Lamb shift.

The main point is theoretical calculation of value

$$\left(\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2})\right).$$

Using this value it is possible

- (i) to evaluate data included 1s and 2s Lamb shift both
- (ii) to re-calculate 1s shift to 2s or 2s to 1s.

Lower as an instance we are going to consider how to obtain 1s from 2s. We will consider only Hydrogen, but this difference has the same expression also for Deuterium and Muonium. Using Deuterium Lamb shift measurement, Isotop shift of $2s \rightarrow 1s$ and Hydrogen 2s-1s measurement it is possible to obtain Rydberg constant without any nuclear corrections calculations or measurements.

2 1s Lamb shift from 2s

Recently a new result for the two-photon transition $2s \rightarrow 1s$ in the Hydrogen has been obtained [1]. To determinate the Rydberg constant from this transition frequency it has to know the Lamb shift of the ground level. Neither the best direct experimental results nor theoretical one are capable to be used without precision lowering.

The most precise value can be evaluated from the follow equation [2, 3]

$$\begin{aligned}\Delta E_L(1s_{1/2}) = & \left(\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2})\right)_{th} \\ & + 8 \left(E(2s_{1/2}) - E(2p_{1/2})\right)_{exp} \\ & + 8 \left(\Delta E_L(2p_{1/2}) + \Delta E_{BG}(2p_{1/2})\right)_{th},\end{aligned}\tag{1}$$

where items with the indexes *th* and *exp* should be obtained theoretically and experimentally, respectively, and $\Delta E_{BG}(2p_{1/2})$ is the known correction of Barker and Glover, arisen from the Breit equation ([4]).

3 Bethe logarithm and natural relativistic parameter

As it is known, the Bethe logarithm is

$$\ln k_0(n, l = 0) \approx 3.$$

That means that main contribution to it is due to continuous spectrum states with energies about $k_0(n, l = 0) \cdot Ry$ and with momenta like

$$\sqrt{k_0(n, l = 0)Z\alpha m} \approx (4.5 \div 4)Z\alpha m,$$

where we use relativistic units: $\hbar = c = 1$.

So the natural relativistic parameter, which is due to expansion of the "Dirac-Bethe logarithm" is $\approx 4Z\alpha$. In some meaning the natural logarithm for Lamb shift is

$$\ln \frac{1 + k_0(n, l = 0)(Z\alpha)^2}{k_0(n, l = 0)(Z\alpha)^2}.$$

That is the reason, why there are large numerical values in a one-loop self-energy contribution.

But in this case the momenta of integration are numerically larger than atomic momentum and hence a large part of contribution could be proportional to square of wave function in the origin (i. e. to factor δ_{l0}/n^3).

4 The main advantages of this calculation

(i) Contributions of the order $\alpha^2(Z\alpha)^5m$ to the Lamb shift are not obtained, but they are equal to zero for $2p_{1/2}$ level and for s-states difference in eq.(1).

(ii) Corrections in the order $\alpha^3(Z\alpha)^4m$ are known only for $\underline{2p_{1/2}}$ state, but they are no corrections in this order to the s-states difference.

(iii) The precision of determination of $\alpha(Z\alpha)^6 m$ and $\alpha(Z\alpha)^7 m$ depends on the extrapolation procedure. There are some large numerical cancellation of their values in the s-states difference and their values in the $2p_{1/2}$ state are much smaller than s-state ones, and having the smaller values some better approximations and higher precision can be obtained. Some higher order corrections in higher than $\alpha(Z\alpha)^7 m$ order are able to be canceled and extrapolation equation can include fewer term and use fewer numerical results.

(iv) There is discrepancy between two proton charge radius measurements (see, e. g., [5, 6]), but the nuclear size corrections are also equal to zero in the s-difference and in the p-state energy.

(v) Some contributions of higher order can be important. The leading correction is [7]

$$\delta E_L^{cub}(ns) = -\frac{8}{27} \frac{\alpha^2(Z\alpha)^6 m}{\pi^2 n^3} \ln^3 \frac{1}{(Z\alpha)^2}, \quad (2)$$

or -3.6 kHz for 2s, -29 kHz for 1s Lamb shift.

The cube logarithm term is canceled for the s-difference and there are neither cube nor square ones in the p-level energy expression.

5 General expression of s-state Lamb shift difference

See e. g. [8].

$$\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2}) = \frac{\alpha(Z\alpha)^4}{\pi} \frac{m_R^3}{m^2} \quad (3)$$

$$\left\{ -\frac{4}{3} \ln \frac{k_0(1s)}{k_0(2s)} \left(1 + Z \frac{m}{M} \right)^2 + (Z\alpha)^2 \right.$$

$$\left. \left((4 \ln 2 - \frac{197}{60}) \ln \frac{1}{(Z\alpha)^2} + (\frac{4}{15} \ln 2 + \frac{1}{140}) + G_s(Z\alpha) \right) \right\}$$

$$-\frac{7}{3} \frac{(Z\alpha)^5}{\pi} \frac{m_R^3}{mM} \left(\frac{3}{2} - 2 \ln 2 \right)$$

$$+\frac{\alpha^2(Z\alpha)^6 m}{\pi^2} \left(\ln^2 \frac{1}{(Z\alpha)^2} A_{262} + O(\ln \frac{1}{(Z\alpha)^2}) \right),$$

where $\ln k_0(nl)$ is the Bethe logarithm, G_s is one-loop self-energy correction in the order $\alpha(Z\alpha)^6 m$ and higher and A_{262} is the leading logarithmic two-loop correction coefficient.

6 General expression of the $2p_{1/2}$ state Lamb shift

$$\Delta E_L(2p_{1/2}) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{m_R^3}{m^2} \left\{ -\frac{4}{3} \ln k_0(2p) \left(1 + Z \frac{m}{M} \right)^2 \right. \quad (4)$$

$$\left. + (Z\alpha)^2 \left(\frac{103}{180} \ln \frac{1}{(Z\alpha)^2} - \frac{9}{140} + G_{2p_{1/2}}(Z\alpha) \right) \right\}$$

$$-\frac{7}{3} \frac{(Z\alpha)^5}{8\pi} \frac{m_R^3}{mM} \left(\frac{1}{6} \right) + \frac{(Z\alpha)^4}{8} \frac{m_R^2}{m} \left(-\frac{1}{3} \right) \left(\frac{1}{2} \frac{\alpha}{\pi} - 0.3285 \left(\frac{\alpha}{\pi} \right)^2 + 1.18 \left(\frac{\alpha}{\pi} \right)^3 \right)$$

$$+ O\left(\frac{\alpha^2(Z\alpha)^6 m}{\pi^2} \ln \frac{1}{(Z\alpha)^2} \right) + O\left(\frac{(Z\alpha)^6 m^2}{M} \right) +,$$

where $\ln k_0(nl)$ is the Bethe logarithm, $G_{2p_{1/2}}$ is one-loop self-energy corrections in the order $\alpha(Z\alpha)^6 m$ and higher.

7 One-loop corrections

To evaluate corrections new numerical results of one-loop self-energy contribution to the Lamb shift [10] have been used. After subtracting all known contributions the rest can be extrapolated to $Z = 1$. We expect that corrections in the order of $\alpha(Z\alpha)^7 \ln(Z\alpha)m$ are proportional to value δ_{l0}/n^3 and our extrapolation equation for G_s and $G_{2p_{1/2}}$

$$G_{nlj}(Z\alpha) = A_{60}(nlj) + (Z\alpha) A_{70}(nlj). \quad (5)$$

leads to results [2, 3]

$$G_s(\alpha) = 0.865(21) \quad (6)$$

and

$$G_{2p_{1/2}}(\alpha) = -0.936(14). \quad (7)$$

The value of eq.(6) is in agreement with one of [9].

8 Two-loop corrections

The leading two-loop contribution to the s-state difference in eq.(1) leads from the follow expression

$$\delta E_L(nlj) = \langle nljm | \Sigma_C(E_{nlj}) \overline{G}_C(E_{nlj}) \Sigma_C(E_{nlj}) | nljm \rangle, \quad (8)$$

where $\Sigma_C(E)$ is one-loop self-energy operator of an electron in the Coulomb field in the Fried-Yennie gauge, $|nljm\rangle$ and E_{nlj} are wave functions and energies in the Dirac Atom of the Hydrogen, and $\overline{G}_C(E)$ is reduced Coulomb Green function of an electron.

The result of evaluation of eq.(8) in logarithmic approximation is [11, 3]

$$\delta E_L(1s_{1/2}) - 8\delta E_L(2s_{1/2}) = \frac{\alpha^2(Z\alpha)^6 m}{\pi^2} \ln^2 \frac{1}{(Z\alpha)^2} A_{262},$$

where

$$A_{262} = -\frac{8}{9} (3 - 2 \ln 2). \quad (9)$$

9 Corrections to the $2p_{1/2}$ -level life-time

The most precise experimental result of the Lamb splitting of the Hydrogen levels $n=2$ can be obtained from measurements [12] of ratio of this splitting and the radiative width of $2p_{1/2}$ level.

The main contribution to the life time of $2p_{1/2}$ state is due to the dipole transition and in this approximation the width of the level has form

$$\Gamma_0 = \frac{4\omega^3}{3} |\mathbf{d}_{12}|^2, \quad (10)$$

where ω is the $2p \rightarrow 1s$ transition frequency, and \mathbf{d}_{12} is the dipole matrix element.

Relativistic corrections in order $(Z\alpha)^2$ have been obtained in [12]. Later in [13] some contributions in the order $\alpha(Z\alpha)^2$ have been also considered. The result of these works is

$$\Gamma = \Gamma_0 \left(1 + (Z\alpha)^2 \left(\ln \frac{9}{8} - \frac{32\alpha}{3\pi} \left(\ln \frac{1}{(Z\alpha)^2} - 2.34 \right) \right) \right). \quad (11)$$

Considering radiative corrections to eq.(10) it should be mentioned that there are two kinds of ones there. A part of corrections is due to transition frequency shifts (i. e. Lamb shifts of $1s_{1/2}$ and $2p_{1/2}$ levels) and the other arises from the dipole matrix element. One can see that the first of them leads to eq.(11), which really includes only a part of $\alpha(Z\alpha)^2$ corrections.

The other are evaluated by me [3, 14] in logarithmic approximation. In the Fried-Yennie gauge the terms with $\ln Z\alpha$ can originate only from $1s_{1/2}$ wave function correction

$$\delta\psi_{1s}(\mathbf{r}) = \sum_{q \neq 1} \psi_{qs}(\mathbf{r}) \frac{\langle qs | \Sigma_C^{(1)} | 1s \rangle}{E_{1s} - E_{qs}}, \quad (12)$$

but not from dipole operator or $2p_{1/2}$ wave function ones. In this equation $\psi(r)$ is Coulomb-Schrödinger wave function in the coordinate representation and the sum has to be done over all discrete and continuous states.

The logarithmic contribution to the width is

$$\delta\Gamma_{wf} = \Gamma_0 \cdot \frac{4}{3\pi} \alpha (Z\alpha)^4 m \ln \frac{1}{(Z\alpha)^2} \sum_{q \neq 1} \left(\frac{\psi_{qs}(0)}{\psi_{1s}(0)} \right) \frac{1}{E_{1s} - E_{qs}} 2 \frac{\mathbf{d}_{q2}}{\mathbf{d}_{12}}, \quad (13)$$

where the sum is over every discrete and continuous states.

After evaluation of eq.(13) we obtain for whole radiative correction

$$\delta\Gamma_{rad} = -\Gamma_0 \cdot \frac{16}{3\pi} \alpha (Z\alpha)^2 \ln \frac{1}{(Z\alpha)^2} (0.49158...). \quad (14)$$

This result has been also obtained analytically¹ from eq.(12)

$$\delta\Gamma = -\Gamma_0 \cdot \frac{16}{3\pi} \alpha (Z\alpha)^2 \ln \frac{1}{(Z\alpha)^2} \left(2 - \frac{1}{2} \left(\ln \frac{1}{2} + \frac{1}{16} + \frac{8}{3} \right) \right). \quad (15)$$

10 Results for the atomic Hydrogen

Let us consider items in eq.(1). Using the results above we find for theoretical contributions to this equation

$$\Delta E_L(1s_{1/2}) - 8\Delta E_L(2s_{1/2}) = -187.237(8) \text{ MHz} \quad (16)$$

and

$$\Delta E_L(2p_{1/2}) = -12.8385(15) \text{ MHz}. \quad (17)$$

The two-logarithmic two-loop corrections of eq.(9) to eq.(16) lead to value of -14.2 kHz . Uncertainties arisen from one-logarithmic ones are estimated as 8 kHz for eq.(16) and 1 kHz for eq.(17). The other part of theoretical uncertainties is due to recoil corrections in order $(Z\alpha)^6(m/M)m$. The uncertainty estimates is 1 kHz for eq.(17), but the contribution to eq.(16) is zero [15]. That is why the uncertainty for s-state difference is smaller than in our works [3, 2, 11]

¹The analytical expression was first obtained by K. Packucki (unpublished) which use an absolutly different way of calculation.

According our evaluation above the highest precision result for the $n = 2$ Lamb splitting from measurements [12] differs from the original result [12] and from reevaluated one [13], both.

New life time of $2p_{1/2}$ is now [3, 14]

$$\tau_{2p_{1/2}} = 1.5961887(15) \cdot 10^{-9} \text{ sec}, \quad (18)$$

where the logarithmic correction eq.(14) leads to $-5.1 \cdot 10^{-15} \text{ sec}$ and our estimate of non-logarithmic $\alpha(Z\alpha)^2$ contributions is $1.5 \cdot 10^{-15} \text{ sec}$.

Using an experimental value of product the life time and the $n = 2$ Lamb splitting measured in work [12] a new value of splitting is obtained [3, 14] as

$$L(2s_{1/2} - 2p_{1/2}) = 1057.8576(21) \text{ MHz}, \quad (19)$$

where the shift from result of [12] without any radiative corrections is -2.91 kHz and our estimate of non-logarithmic contribution is 1 kHz . This result is in agreement with direct experimental values and with theoretical one.

After summarizing items of eq.(1) according eq.(16), eq.(17) and eq.(19) we obtain

$$\Delta E_L(1s_{1/2}) = 8172.934(22) \text{ MHz}. \quad (20)$$

This result has higher precision than both of direct experimental and theoretical ones.

A more detailed evaluation is presented in [3] and [2, 11, 14].

Let us discuss other Lamb shift values. Using Lundeen & Pipkin result [16] we can obtain

$$\Delta E_L(1s_{1/2}) = 8172.833(73) \text{ MHz}, \quad (21)$$

and Hagley & Pipkin one [17] leads to

$$\Delta E_L(1s_{1/2}) = 8172.785(97) \text{ MHz}. \quad (22)$$

Using the difference eq.(16) we obtain from Garching measurement [18]

$$\Delta E_L(1s_{1/2}) = 8172.857(129) \text{ MHz}, \quad (23)$$

where 128 kHz leads from measurement and theoretical uncertainty from eq.(16) is only 13 kHz.

We can re-evaluate results on $8s/d \rightarrow 2s$ [19] and $2s \rightarrow 1s$ [1] two-photon transitions frequencies to obtain result for 1s Hydrogen Lamb shift

$$\Delta E_L(1s_{1/2}) = 8172.786(118) \text{ MHz}. \quad (24)$$

All results for Hydrogenic atom are collected in the Table 1 and on the Fig. 1. Some discussion of theoretical results is done lower with considering of similar corrections to Muonium hyperfine splitting. We have incule into the Table 1 and the Fig. 1 theoretical results with both proton radii and with and without two-loop corrections. Result without $\alpha^2(Z\alpha)^5 m$ [21] is included only because "with $\alpha^2(Z\alpha)^5 m$ [21]"-result is in huge disagreement with the Lamb shift of the Helium-Ion [20]. It should be also mentioned that the large two-loop result is in agreement with measurements of hydrogen-like ions of the Phosphorus [22] and the of Sulfur [23].

The two-loop corrections in normalization of [21] are

$$H_{50}^P = -55(23), \quad \text{from [23]}, \quad (25)$$

and

$$H_{50}^S = -62(36), \quad \text{from [22]}, \quad (26)$$

and theoretical values [21, 7] are

$$H_{50}(Z) = B_{50} (1 \pm (Z\alpha)) - \frac{8}{27}(Z\alpha) \ln^3 \frac{1}{(Z\alpha)^2} (1 \pm 1), \quad (27)$$

or

$$H_{50}(Z = 15) \approx H_{50}(Z = 16) = -24(4). \quad (28)$$

The average value from the Chlorine and Argon is $H_{50}^{Cl\&Ar} = -85(74)$.

In this language the Helium disagreement is

$$H_{50}^{He} = 0.2(4.6), \quad \text{from [20]} \quad (29)$$

instead a theoretical value $H_{50}(Z = 2) = -27(3)$. The Lithium experimental value is $H_{50}^{Li} = 61(52)$.

Some theoretical discussion of two-loop corrections is done some later.

11 Results for the Deuterium

We could also work for Deuterium s-state difference. The result is

$$\Delta E_L(1s_{1/2}) - 8\Delta E_L(1s_{1/2}) = -187.229(8) \text{ MHz}, \quad (30)$$

or 8 kHz higher than for Hydrogen and from Garching experiment [18] it leads to

$$\Delta E_L(1s_{1/2}) = 8183.905(224) \text{ MHz}, \quad (31)$$

where uncertainty from eq.(30) is only 13 kHz.

We can also obtain the Deuterim ground state Lamb shift using equations eq.(16) and eq.(30) for s-state differences of Hydrogen and Deuterium, the (1s-2s)-isotope shift measurement [24] and some result of the 1s Hydrogen Lamb shift. The value of eq.(23) leads to Garching-Garching result

$$\Delta E_L(1s_{1/2}) = 8183.967(132) \text{ MHz}, \quad (32)$$

which is in a good agreement with the direct Garching value eq.(31).

12 Result for the Muonium Lamb shift

The result for the Muonium is

$$\Delta E_L(1s_{1/2}) - 8\Delta E_L(1s_{1/2}) = -187.348(8) \text{ MHz}. \quad (33)$$

13 $\alpha^2 Ry$ term and future measurements

Real experimental values in so-called *Lamb shift* measurements are rather Dirac correction to the Schödinger energies. Dirac corrections for Garching 1s Lamb shift experiment is 3928.707(12) MHz where the uncertainty is experimental one.

The Paris-Garching result is

$$f(8d - 2s) - \frac{5}{16}f(2s - 1s) = 4187.518(18) \text{ MHz}, \quad (34)$$

where frequencies are double frequencies of two-photon transitions and uncertainty items are 10 kHz (Paris) and 14 kHz (Garching).

Using these values one can obtain some fine structure constant values with uncertainties like $(1.5 \div 2) \cdot 10^{-6}$. But at least the Garching results are going to be improved and to have uncertainties reduced by factor 10 [25]. The Paris results are also going to be improved [26]. That means that to calculate Dirac corrections should know the fine structure constant with high precision.

14 The fine structure constant

There are a lot of way to determinate the fine structure constant. The main experimental values are presented in the Table 2 and on the Figs. 2. There are two kinds of values there. Some lead from electrical measurements and is due to electrical standards. The other have no connections with them. The main non-electrical values are results from investigations of the electron anomalous magnetic moment [27, 28]

$$\alpha_{AMM}^{-1} = 137.0359922(9), \quad (35)$$

and one from the neutron de Broglie's wave length measurements [29]

$$\alpha_n^{-1} = 137.0360105(54). \quad (36)$$

The value from the photon recoil result for h/M_{Cs} includes only statistical error [30].

We are considered non-electrical results later. A detailed discussion for electrical ones is presented in a review of Cohen and Taylor [31].

Equations to determinate α^{-1} are follow:

$$\alpha_{g-2}^{-1} = \frac{a_e}{A_2 + \frac{\alpha}{\pi}A_4 + \dots}, \quad (37)$$

$$\alpha_{Mu}^{-2} = \frac{16}{3}(Ry \cdot c)(\mu_\mu/\mu_p)(\mu_p/\mu_B)(\Delta\nu_{hfs}^{-1})(1 + \dots), \quad (38)$$

$$\alpha_n^{-2} = \frac{(m_n \cdot c/h)}{2 \cdot Ry \cdot (m_n/m_e)}, \quad (39)$$

$$\alpha_{Cs}^{-2} = \frac{(M_{Cs} \cdot c/h)}{2 \cdot Ry \cdot (M_{Cs}/m_e)}, \quad (40)$$

$$\alpha_{R_K, K_\Omega}^{-1} = (2 \cdot R_K / \mu_0 c) K_\Omega, \quad (41)$$

$$\alpha_{\gamma'_p, K_\Omega}^{-2} = c \cdot (4 \cdot Ry \cdot \gamma'_p)^{-1} (\mu'_p / \mu_B) (2e/h) K_\Omega^{-1}, \quad (42)$$

$$\alpha_{\gamma'_p, R_K}^{-3} = R_K (2 \cdot \mu_0 \cdot Ry \cdot \gamma'_p)^{-1} (\mu'_p / \mu_B) (2e/h), \quad (43)$$

where K_Ω is a ratio of the SI Ohm and the BIPM Ohm, R_K is the Klitzing constant which is expected to be equal to the Quantum Hall resistance, γ'_p is the giromagnetic ratio of the proton measured in the water by low-field method, and all electrical values are measured in the BIPM units. The QED corrections is presented as "...".

15 The anomalous magnetic moment of electron

The theoretical expression of anomalous magnetic moment of electron has form

$$a_e = A_2 \frac{\alpha}{\pi} + A_4 \frac{\alpha^2}{\pi^2} + A_6 \frac{\alpha^3}{\pi^3} + A_8 \frac{\alpha^4}{\pi^4} + \delta a, \quad (44)$$

where electron-photon contributions are [28, 32] (see also a review [33])

$$A_2(e) = 0.5, \quad (45)$$

$$A_4(e) = -0.328478965..., \quad (46)$$

$$A_6(e) = 1.17611(42), \quad (47)$$

$$A_8(e) = -1.424(138). \quad (48)$$

Heavy leptons contributions are

$$\delta a(\mu) = 2.80410^{-12}, \quad (49)$$

$$\delta a(\tau) = 0.01010^{-12}. \quad (50)$$

Non-QED corrections are also known

$$\delta a(had) = 1.6(2)10^{-12}, \quad (51)$$

$$\delta a(weak) = 0.0510^{-12}. \quad (52)$$

That leads to result

$$a_e = 1159652140(27.1)(5.3)(4.1) \cdot 10^{-12}, \quad (53)$$

where we use α from *so-called* the Quantum Hall effect measurement (see Table 2)

$$\alpha_{R_K, K_\Omega}^{-1} = 137.0359979(32). \quad (54)$$

The items of uncertainties are 27.1 from α , 5.3 from A_6 and 4.1 from A_8 . The results for the fine structure constant is

$$\alpha_{g-2}^{-1} = 137.03599222(63)(48)(51) \quad (55)$$

where two first uncertainties are due A_6 and A_8 , the last arisen from the measurements [27]

$$a_{e-} = 1159652187.9(4.3)10^{-12} \quad (56)$$

and

$$a_{e+} = 1159652188.4(4.3)10^{-12}. \quad (57)$$

16 The Muonium hyperfine splitting

The main contribution to the Muonium ground state hyperfine splitting can be written as

$$\nu_F = \frac{16}{3}(Z\alpha)^2 c R_\infty \frac{m}{M} \left[\frac{m_R}{m} \right]^3 (1 + a_\mu). \quad (58)$$

The quantum electrodynamics part of the HFS interval is [8, 35, 36, 37, 38]

$$\Delta\nu(QED) = \nu_F \left(1 + \frac{3}{2}(Z\alpha)^2 + \frac{17}{8}(Z\alpha)^4 + \dots \right) \quad (59)$$

$$+ \nu_F \left(a_e + \alpha \left\{ (Z\alpha) \left(\ln 2 - \frac{5}{2} \right) \right. \right.$$

$$\left. - \frac{8(Z\alpha)^2}{3\pi} \ln(Z\alpha) \left[\ln(Z\alpha) - \ln 4 + \frac{281}{480} \right] \right.$$

$$\left. + \frac{(Z\alpha)^2}{\pi} (15.38 \pm 0.29) \right\} + \frac{\alpha^2(Z\alpha)}{\pi} D)$$

$$+ \frac{\nu_F}{1 + a_\mu} \left(- \frac{3Z\alpha}{\pi} \frac{mM}{M^2 - m^2} \ln \frac{M}{m} \right.$$

$$\left. + \frac{\gamma^2}{mM} \left[2 \ln \frac{1}{2Z\alpha} - 6 \ln 2 + \frac{65}{18} \right] \right)$$

$$+ \frac{\nu_F}{1 + a_\mu} \frac{\alpha(Z\alpha)}{\pi^2} \frac{m}{M} \left(- 2 \ln^2 \frac{M}{m} + \frac{13}{12} \ln \frac{M}{m} \right.$$

$$\left. + \frac{21}{2} \zeta(3) + \frac{\pi^2}{6} + \frac{35}{9} \right)$$

$$+ \nu_F \frac{\alpha^2(Z\alpha)}{\pi^3} \frac{m}{M} \left(- \frac{4}{3} \ln^3 \frac{M}{m} + \frac{4}{3} \ln^2 \frac{M}{m} + \dots \right),$$

where D is two-loop coefficient. Two loop corrections and higher order contributions are considered in next sections.

Non-QED corrections are known [39]

$$\Delta\nu(\text{strong}) = \nu_F \frac{\alpha(Z\alpha)}{\pi^2} \frac{m}{M} (2.15 \pm 0.14), \quad (60)$$

and [40, 41]

$$\Delta\nu(\text{weak}) \simeq 0.065 \text{ kHz}. \quad (61)$$

17 Two-loop corrections

Two-loop corrections of absolute order $\alpha^2(Z\alpha)^5$ are induced by six gauge-independent sets of diagrams (see e.g.[42, 50]) which are presented in Fig. 3.

The contribution to the HFS interval in the Muonium ground state are:

$$\Delta\nu^a = \nu_F \frac{\alpha^2(Z\alpha)}{\pi} \frac{36}{35}, \quad [42] \quad (62)$$

$$\Delta\nu^b = \nu_F \frac{\alpha^2(Z\alpha)}{\pi} \left(\frac{224}{15} \ln 2 - \frac{38}{15} \pi - \frac{118}{225} \right), \quad [42] \quad (63)$$

$$\Delta\nu^c = \nu_F \frac{\alpha^2(Z\alpha)}{\pi} \left(-\frac{4}{3} \mathcal{L}^2 - \frac{20\sqrt{5}}{9} \mathcal{L} - \frac{64}{45} \ln 2 \right. \quad (64)$$

$$\left. + \frac{\pi^2}{9} + \frac{1043}{675} + \frac{3}{8} \right), \quad [42]$$

where

$$\mathcal{L} = \ln \frac{1 + \sqrt{5}}{2},$$

$$\Delta\nu^d = \nu_F \frac{\alpha^2(Z\alpha)}{\pi} (-0.310\,742\dots). \quad [43] \quad (65)$$

These results have been independently checked in the work [45].

The light-by-light scattering diagram led to slight disagreement between [44] and [45], which was due to the misprint on an intermediate state of work [44] (see [46]). The final result is

$$\Delta\nu^{(e)} = \nu_F \frac{\alpha^2(Z\alpha)}{\pi} (-0.47251\dots). \quad [44, 46], \quad [45] \quad (66)$$

The result of the sixth set calculation is

$$\Delta\nu^f = \nu_F \frac{\alpha^2(Z\alpha)}{\pi} (-0.63(4)). \quad [45] \quad (67)$$

This figure was obtained in Feynman gauge. An other way is used in works [47, 48], where the first nine diagram of this nineteen-diagram set (see Fig.4) were evaluated. Evaluation is done in Fried-Yennie gauge. The result for sum of three first diagram of [45] is in agreement with earlier result of [47]. The calculations in Fried-Yennie gauge are going to be completed [49].

18 Two-loop corrections. Comparison: HFS and Lamb shift

The contributions to HFS interval can be written as

$$\Delta\nu_{hfs} = \nu_F \frac{\alpha^2(Z\alpha)}{\pi n^3} \left(\frac{8}{\pi^2} \int_0^\infty \frac{d|\mathbf{k}|}{\mathbf{k}^2} F_{hfs}(\mathbf{k}) \right), \quad (68)$$

where $F_{hfs}(\mathbf{k})$ is some substructured form-factor, normalized as 1 in the skeleton diagram. The subtraction need to exclude the lower order corrections.

The contributions to Lamb shift can be written as

$$\Delta\nu_{Ls} = \frac{\alpha^2(Z\alpha)^5 m}{\pi^2 n^3} \left(-\frac{16}{\pi} \int_0^\infty \frac{d|\mathbf{k}|}{\mathbf{k}^4} F_{Ls}(\mathbf{k}) \right), \quad (69)$$

where $F_{hfs}(\mathbf{k})$ is some substructured form-factor, normalized also as 1 in the skeleton diagram. The subtraction need also to exclude the lower order corrections.

To compare contribution let us consider integrals

$$I = \frac{8}{\pi} \int_0^\infty \frac{d|\mathbf{k}|}{\mathbf{k}^{2r}} F_{Ls}(\mathbf{k}), \quad (70)$$

where $r_{hfs} = 1$ and $r_{Ls} = 2$. In Table 3 we include the numerical results for all I , and asymptotic behaviours of integrand $F(\mathbf{k})/\mathbf{k}^{2r}$. For non-leading degree of k no logarithmic factors are presented.

Taking into account that the integration of logarithm should lead to higher result, we can expect that Lamb shift contribution of a, b, c, d, e should be smaller, but f-corrections should be larger. Really such estimate could be not correct if constant is quite large, as it is in two-loop vacuum polarization (Fig. 3c). But in this case we have additional reason to expect lower contribution to the Lamb shift. The k^4 -term in low-energy expansion of polarization is product of k^2 term and $k^2/(2m)^2$ and some constant, which is smaller than 1. So after subtracting the low-energy part of integral should be smaller.

Considering the Pachucki result [21], it should be mentioned that integrand has correct asymptotic behaviours for both low and high momenta, and integrating the low-energy asymptotics from 0 to the electron mass and the high-energy one from twice electron mass to the infinity we can obtain

$$I_{Ls}^f = (7 \div 14) + 3, \quad (71)$$

where to estimate uncertainty we use at low-energy

$$\left(\ln 1/k - 1 \pm 1 \right) \quad (72)$$

instead $\ln 1/k$.

We expect that the asymptotics is rather some over-estimate of the real integrand and so a naturel magnitude of this integral is 10 ± 5 .

In work [21] the Fried-Yennie gauge was used for the first three diagram of Fig 4, and Feynman gauge for the other graphs. The result of three first diagram of [21] is in agreement with earlier result of [47], where the first nine diagram of this nineteen-diagram set (see Fig.4) were evaluated. Evaluation is done in Fried-Yennie gauge. The calculations are going to be completed [48, 49].

19 Higher order logarithmic corrections

The leading logarithmic radiative-recoil corrections are known

$$\delta\nu^{(1)}(rad - rec) = \nu_F \frac{\alpha^2(Z\alpha)}{\pi^3} \frac{m}{M} \left(-\frac{4}{3} \ln^3 \frac{M}{m} + \frac{4}{3} \ln^2 \frac{M}{m} \right) \quad (73)$$

The cubic term was obtained [55], and square of recoil logarithm was calculated in [56]. Some additional discussion on light-by-light scattering contribution was done in [57], and on diagrams without closed electron loop in [58]. The contributions of these recoil terms are -0.04 and +0.01 kHz.

More valuable corrections include low-energy logarithm. The well-known leading corrections in order $\alpha(Z\alpha)^2$ can be obtained without using any expansion of the mass ratio m/M . The result is [7]

$$\delta\nu^{(2)}(leading) = \nu_F \frac{\alpha}{\pi} (Z\alpha)^2 \left(-\frac{8}{3} \left(\frac{m_R}{m} \right)^2 \ln^2 Z\alpha \right). \quad (74)$$

Hence, a new radiative-recoil correction is [7]

$$\delta\nu^{(2)}(rad - rec) = \nu_F \frac{\alpha}{\pi} (Z\alpha)^2 \frac{m}{M} \left(+\frac{16}{3} \ln^2 Z\alpha \right), \quad (75)$$

or 0.34 kHz. The contribution of the electron anomalous magnetic moment is [7]

$$\delta\nu(rad) = \nu_F \left(\frac{\alpha}{\pi} \right)^2 (Z\alpha)^2 \left(-\frac{4}{3} \ln^2 Z\alpha \right), \quad (76)$$

or -.04 kHz.

Taking into account the recoil correction to the Lamb shift instead the pure radiative contribution we can also obtain a recoil contribution [7]

$$\delta\nu^{(1)}(rec) = \nu_F \frac{(Z\alpha)^3}{\pi} \frac{m}{M} \left(-\frac{2}{3} \ln^2 Z\alpha \right), \quad (77)$$

or -.04 kHz. However, there are another logarithmic contribution. This can be evaluated from the leading recoil contribution but with the Dirac wave functions [59, 60]

$$\delta\nu^{(2)}(rec) = \nu_F \frac{(Z\alpha)^3}{\pi} \frac{m}{M} \left(-3 \ln \frac{M}{m} \ln \frac{1}{Z\alpha} \right), \quad (78)$$

the magnitude of corrections is -.21 kHz

The leading higher order correction to the Lamb shift is [7]

$$\delta E_L^{cub}(ns) = -\frac{8}{27} \frac{\alpha^2 (Z\alpha)^6 m}{\pi^2 n^3} \ln^3 \frac{1}{(Z\alpha)^2}, \quad (79)$$

or -3.6 kHz for 2s, -29 kHz for 1s Lamb shift. This corrections is due to the same set of diagram (Fig. 3f) as conntribution [9] and it is about 9% of it.

The part of logarithm square corrections arises from set of Fig 3c.

$$\delta E_L^{sq,f}(ns) = \frac{4}{45} \frac{\alpha^2 (Z\alpha)^6 m}{\pi^2 n^3} \ln^2 \frac{1}{(Z\alpha)^2}, \quad (80)$$

or .88 kHz for 1s Lamb shift. This corrections is about 4% of $\alpha^2 (Z\alpha)^5 m$ contribution of diagram Fig. 3c.

20 The Muonium and the fine structure constant

To determinate the fine structure constant we sould re-wrire the theoretical equation using only values which can be measured

$$\Delta\nu(th) = \frac{16}{3} \alpha^2 cRy \left(\frac{\mu_\mu}{\mu_p} \frac{\mu_p}{\mu_B} \right) \left[\frac{m_R}{m} \right]^3 \{1 + \dots\}, \quad (81)$$

where the main problem is due to the muon magnetic moment. All known result are [63, 62] and references there in good agreement but they have not quite high precision (Fog 5). The new result is going to have uncertainty reduced by factor like $3 \div 5$ and relative uncertainty for α will be $(3 \div 5) \cdot 10^{-8}$.

The final theoretical result for the HFS interval is

$$\Delta\nu(th) = 4463303.6(13)(2) kHz, \quad (82)$$

where we use α from the anomalous magnetic moment of electron, and uncertainty items due to the muon magnetic moment (average value of [63, 62] and theoretical calculations (numerical error of integration in [34] and an estimate of higher-order uncalculated contributions).

The highest precision experimental result is

$$\Delta\nu(exp) = 4463302.88(16)kHz. \quad (83)$$

21 The neutron de Broglie's wave length

The neutron result of α is

$$\alpha_n^{-1} = 137.035993(27). \quad (84)$$

It obtained from experimental results for the Rydberg constant, the neutron-electron masses ratio, the neutron de Broglie's wave length (h/m_nv) [29], which was measured in unity of some known crystal lattice spacing, and the neutron velocity v [29]:

$$\alpha_n = \sqrt{\frac{cRy(m_n/m_e)}{(m_nv/2h)v}}. \quad (85)$$

This result is in disagreement with the one from the electron anomalous magnetic moment

$$\alpha_{AMM}^{-1} = 137.0359922(9). \quad (86)$$

The use of a crystal leads to a direct connection between the Silicon spacing measured and the fine structure measurements. We can use eq.(85) and eq.(86) and obtain indirect result for the Silicon spacing

$$d_{220}(indirect) = 192015.617(10) \text{ fm} \quad (87)$$

instead PTB direct result [64]

$$d_{220}(direct) = 192015.568(12) \text{ fm}. \quad (88)$$

Recently a new result for the spacing have been obtained [65]. It is equal to

$$d_{220} = 192015.569(6) \text{ fm}. \quad (89)$$

So we can expect that after some re-evaluating, the neutron result for the fine structure constant should have the same value but twice smaller

uncertainty (its relative value will be $\approx 2 \cdot 10^{-8}$). The one for the anomalous moment is three times smaller, but disagreement is $\approx 1.3 \cdot 10^{-7}$.

All values of the Silicon lattice spacing [64, 65, 66] are presented on Fig. 6.

We also expect that some connection between Avogadro constant measurements and the fine structure constant could appear. Generally, the results for α are obtained with higher precision, but there is a factor 6 between uncertainty values. That is because the fine structure constant is connected with square root of the spacing and the Avogadro constant is proportional to inverse cube of it.

The other problem is due the neutron-electron mass ratio. The neutron mass is result of measurements of the proton mass, Deuteron mass and the Deuteron binding energy. For the last value the highest precision result [67] has some not very large disagreement with others [68] (see Fig. 7). The shift of α could be 10^{-8} .

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Table 1. The Lamb shift of the Hydrogen ground state

<i>Ref.</i>	Result
Lundeen & Pipkin	8172.833(73) MHz
Sokolov & Yakovlev (corrected)	8172.934(22) MHz
Hagley & Pipkin	8172.785(97) MHz
Garching	8172.857(129) MHz
Garching & Paris	8172.786(118) MHz
$r_p = .862 fm, \alpha^2(Z\alpha)^5 m$	8172.762(40) MHz
$r_p = .862 fm$	8173.053(40) MHz
$r_p = .805 fm, \alpha^2(Z\alpha)^5 m$	8172.613(40) MHz
$r_p = .805 fm$	8172.904(40) MHz

Table 2. The (inverse) fine structure constant α^{-1}

<i>NN.</i>	Method	Result
1.	$(g_e - 2)$	137.035 9922(9)
2.	$R_K, K_\Omega, NIST$	137.035 9979(33)
3.	$\gamma'_p, R_K, NIST$	137.035 9840(51)
4.	h/m_n	137.0360102(54)
5.	h/M_{Cs}	137.0360876(71)
6.	R_K, K_Ω, NPL	137.0360084(74)
7.	$R_K, K_\Omega, CSIRO/NML$	137.0360093(90)
8.	$\gamma'_p, R_K, VNIIM$	137.035 949(16)
9.	<i>Muhfs</i>	137.036003(20)
10.	<i>Cohen, Taylor, 1992</i>	137.035 9928(9)
11.	<i>CODATA, 1986</i>	137.035 9986(62)
12.	<i>CCE, 1990</i>	137.035 997(27)

Table 3a.
The two-loop values of HFS [42], [43], [44, 46], [45]

$NN.$	a	b	c	d	e	f	tot
I	3.24	5.87	-2.10	-.98	-1.48	-2.0	2.56
$k \ll m$	k^2	1	1	1	k^2	$\log 1/k$	-
$k \gg m$	$\log^2 k/k^2$	$\ln k/k^2$	$\log k/k^2$	$\log k/k^2$	$\log k/k^2$	$1/k^2$	-

Table 3b.
The two-loop values of Lamb shift, [50], [51], [52], [53], [54], [21]

$NN.$	a	b	c	d	e	f	tot
I	.096	-.80	-.96	.11	.19	12.0	10.6
$k \ll m$	1	1	$\log 1/k$	1	1	$\log^2 1/k$	-
$k \gg m$	$1/k^4$	$1/k^2$	$1/k^4$	$1/k^2$	$1/k^4$	$1/k^2$	-